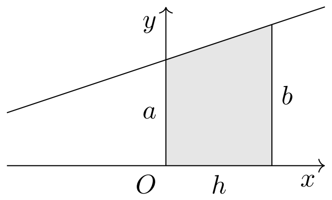
1201. If $\mathrm{f}(x) = \dfrac{1}{1+x}$, prove that $\mathrm{f}^2(x) = \dfrac{x+1}{x+2}$.

1202. By finding the equation of the line and setting up an integral, verify the area formula $A = \frac{1}{2}(a+b)h$ for the right-angled trapezium depicted below.



1203. True or false?

(a) $\dfrac{x^2-1}{x^2+1} = 0$ has two real roots,

(b) $\dfrac{x^2-1}{x^2+x} = 0$ has two real roots,

(c) $\dfrac{x^2-1}{x^2-2} = 0$ has two real roots.

1204. Find an irrational number which is a member of the set $[3.6, 3.61]$.

1205. A graph has equation $y = \dfrac{2x+7}{x+3}$.

(a) Express the graph as $y = a + \dfrac{b}{x+3}$.

(b) Hence, sketch the curve.

1206. A set of 100 straight lines is drawn in a plane, such that no three lines are concurrent and no two lines are parallel. Find the number of intersections.

1207. A differential equation is given as

$$\frac{dy}{dx} = 2x - 3y.$$

A linear solution $y = mx + c$ is suggested.

(a) Write down the value of $\dfrac{dy}{dx}$.

(b) Substitute to show that m and c must satisfy the identity $3mx + m \equiv 2x - 3c$.

(c) Hence, show that the only linear solution to the differential equation is $9y - 6x + 2 = 0$.

1208. Solve the equation $x^5 = 8x^2$.

1209. A uniform solid cube of mass m has vertices $ABCDEFGH$, where $ABCD$ is the horizontal lower face. The cube is supported by vertical forces at A, B and the midpoint of CD.

(a) Explain, using a plan sketch, why the forces at A and B must have the same magnitude.

(b) Find the magnitudes of the forces.

1210. Write down the value of $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\, dx$.

1211. State, with a reason, whether $x = k$, where $k \in \mathbb{R}$ is a constant, intersects the following curves:

(a) $y = x^2 + k + 1$,

(b) $x = y^2 + k + 1$.

1212. A square is undergoing an enlargement. The rate of change of the lengths of its sides is 2 cm/s. Find the rate of change of the lengths of its diagonals.

1213. Determine the number of points (x, y) for which $x, y \in \mathbb{Z}$ and $2x^2 + 2y^2 < 9$.

1214. The Newton-Raphson iteration

$$x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$$

is being used to factorise $\mathrm{f}(x) = x^3 - 5x^2 - 2x - 24$.

(a) Set up the iteration for f.

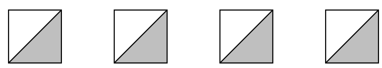
(b) Run the iteration with $x_0 = 0$, and verify that it converges to a root α of f.

(c) Explain why $(x - \alpha)$ must be a factor of $\mathrm{f}(x)$.

(d) Factorise $\mathrm{f}(x)$, using the discriminant to show that no further factorisation is possible.

1215. On one set of axes, sketch the graphs $y = 2^{x-1}$ and $y = 4^{x-1}$, marking any points of intersection.

1216. The four tiles below are placed together, in random orientations, to form a two-by-two square.



(a) Show that there are 13 configurations in which the shading forms one region.

(b) Hence, find the probability that the shading forms one region.

1217. If $\dfrac{d}{dx}(x^2 + 2y) = 10$, find $\dfrac{dy}{dx}$ in terms of x.

1218. A statistician is setting up a hypothesis test for the mean μ of a normal distribution $X \sim N(\mu, \sigma^2)$. Under the assumption of the null hypothesis, an acceptance region (a, b) is calculated, such that

$$\mathbb{P}(a < \bar{X} < b) = 0.99.$$

(a) Is the test one-tailed or two-tailed?

(b) What is the significance level of the test?

1219. A geometric sequence has nth term u_n. Show that, for any constants $p, q \in \mathbb{R}$, the following sequence is also geometric:

$$w_n = pu_n + qu_{n+1}.$$

1220. Either prove or disprove the following statement: "The product of irrational numbers is irrational."

1221. The equations $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, where f and g are linear functions defined over $\mathbb{R}$, have the same solution set S. The equation $\mathrm{f}(x) = \mathrm{g}(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

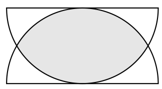
(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

1222. Show that $\displaystyle\int_1^2 \frac{1}{x^2} + \frac{2}{x^3}\,dx = \tfrac{5}{4}$.

1223. A reaction force is so called because it occurs in "reaction" to contact. Explain whether or not it is possible for an object to experience a reaction force while weightless.

1224. Two semicircles, with parallel diameters of length 2, overlap symmetrically as follows:



Show that the unshaded region has area $\sqrt{3} - \frac{1}{3}\pi$.

1225. Find the sum of the first 1000 odd numbers.

1226. The quadratic functions f, g, h have discriminants which are negative, zero, and positive respectively. The functions share no roots. Write down the numbers of roots of the following equations:

(a) $\mathrm{f}(x)\,\mathrm{g}(x) = 0$,

(b) $\mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{h}(x) = 0$,

(c) $\mathrm{g}(x)\big(\mathrm{h}(x)\big)^{-1} = 0$.

1227. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y > x + 1, \qquad y < 1 - x.$$

1228. A sample has $\bar{x} = 1.17$, $\sum x^2 = 227$ and variance 0.9011. Find the number of data in the sample.

1229. The scores a and b on two exams, out of A and B marks respectively, are to be combined into one score X, given out of a hundred. Each mark is to have equal weighting. Find a formula for X.

1230. Prove that a pair of distinct circles cannot have more than two intersections.

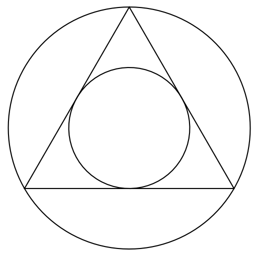
1231. Take the acceleration due to gravity to be 10 ms^{-2} in this question, and neglect air resistance.

A rocket of mass 120 kg is fired vertically into the air from rest. Upon ignition, due to its boosters, it feels a constant force of 1500 N vertically.

(a) Modelling the mass as constant, find the time taken to reach an altitude of 500 m.

(b) At this time, the fuel is exhausted. Determine the speed at which the rocket hits the ground.

1232. On an equilateral triangle, circles C_1 and C_2 are maximally inscribed and minimally circumscribed.



Show that the ratio of areas is 1 : 4.

1233. It is given that $\{A_n\}$ is a quadratic sequence and $\{B_n\}$ is an increasing arithmetic sequence.

Describe the following sequences as quadratic, arithmetic or neither:

(a) $u_n = (A_n)^2$,

(b) $u_n = (B_n)^2$.

1234. A jar contains 6 blue and 4 red counters. From the jar, two counters are taken out at random. Find the probability that these are different colours.

1235. Prove that, for any function f and constant c, the curves $y = \mathrm{f}(x)$ and $y = c - \mathrm{f}(x)$ have the same set of x values at which they are stationary.

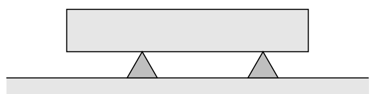
1236. State, with a reason, whether or not the following are valid identities:

(a) $\sin|x| \equiv \sin x$,

(b) $\cos|x| \equiv \cos x$,

(c) $\tan|x| \equiv \tan x$.

1237. A uniform block of mass m kg rests on supports, as depicted. The supports divide the length of the block into sections in the ratio 2 : 3 : 1.



Find the reaction forces at the supports.

1238. Evaluate $\displaystyle\lim_{x\to 2} \frac{2x^2 + 6x - 20}{3x^2 - 7x + 2}$.

1239. Show that the quadratic $2x^2 + 7x - 4$ is a factor of the cubic $2x^3 - 7x^2 - 53x + 28$.

1240. A number is given in standard form as $z = a \times 10^b$.

(a) Write down the set of possible values of a.

(b) Show that $\log_p z$ is linear in b, for any $p > 0$.

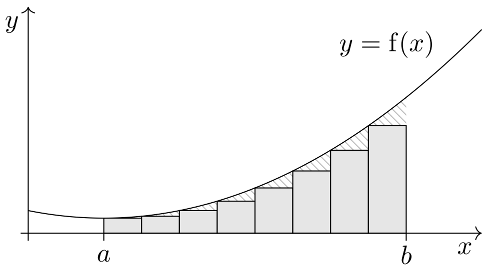
1241. The expression $k - 7x + 6x^2$ has discriminant 121. Find the value of the constant k.

1242. Prove that, if a parallelogram is also a kite, then it is a rhombus.

1243. It is a fundamental result of calculus that

$$\lim_{\delta x \to 0} \sum_a^b \mathrm{f}(x)\, \delta x \equiv \int_a^b \mathrm{f}(x)\, dx.$$

This identity relates to the diagram show below, in which δx is the width of each rectangular strip.



In each case, write down the feature of the diagram represented by the algebraic expression:

(a) $\mathrm{f}(x)$

(b) $\mathrm{f}(x)\, \delta x$

(c) $\sum_a^b \mathrm{f}(x) \delta x$

(d) $\int_a^b \mathrm{f}(x)\, dx$.

1244. Determine the values of the constants a and b for the following to be an identity in x:

$$x(x-2)(x-a) \equiv x^3 + bx^2 + 6x.$$

1245. A stepladder, when erected, has the shape of two sides of an equilateral triangle of side length 240 cm. Without the use of a calculator, show that the ladder will not fit underneath a shelf 2 metres off the ground.

1246. Show that, as $x \to 0$, the gradient of the tangent to $y = \sqrt{x}(x-1)$ increases without bound.

1247. Statements P and Q about hexagon H are:

P : "H is regular."

Q : "H has sides described, tip-to-tail around the perimeter, by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, -\mathbf{a}, -\mathbf{b}, -\mathbf{c}$."

State, with a reason, which, if any, of the following implications hold:

① $P \implies Q$

② $P \impliedby Q$

③ $P \iff Q$

1248. Solve the equation $2 \log_x 2 = 1 - 2 \log_x 3$.

1249. A function F is set up as follows: given a monic quadratic graph $y = \mathrm{f}(x)$ as an input, the output of F is the vertex, as a pair of coordinates. For example,

$$\mathrm{F}\left[y = (x-2)^2 + 5\right] = (2, 5).$$

(a) Find $\mathrm{F}\left[y = x^2 - 2x + 7\right]$.

(b) Find the function g in $\mathrm{F}\left[y = \mathrm{g}(x)\right] = (3, 0)$.

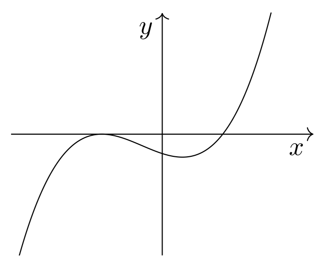
1250. Show that an equilateral triangle of side length 2 will fit exactly inside a circle of radius $\frac{2\sqrt{3}}{3}$.

1251. Evaluate $\sum_{k=1}^{\infty} \frac{3}{10^k}$ as a simplified fraction.

1252. Prove that the area A of a rhombus is given by the formula $A = l^2 \sin\alpha$, where l is side length and α is any interior angle.

1253. Use the Newton-Raphson iteration to determine, to 3sf, the input value for which the reciprocal function and the natural logarithm function give the same output.

1254. Determine whether $y = (x+1)(x-1)^2$ could be the equation generating the following graph:



1255. State, giving a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a real number x:

① $-x^3 = 3$,

② $x^6 = 9$.

1256. "The x axis is tangent to the curve $y = x^4 - x^2 - 6$." True or false?

1257. Simultaneous equations are given as:

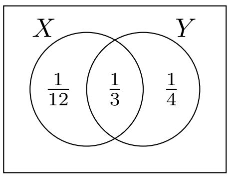
$$\begin{aligned} a+4b+c&=10,\\ a+b+2c&=12,\\ 3a+b+c&=9. \end{aligned}$$

(a) From the first two, show that $3a+7c=38$.
(b) Find another equation linking a and c.
(c) Hence, solve for a, b, c.

1258. In a rugby scrum, two teams are pushing against one another. The two teams have total mass 850 kg and 950 kg respectively. The scrum accelerates constantly from rest, moving 1 metre in 4 seconds. Determine the difference in driving force exerted by the two teams.

1259. A quadratic sequence starts $7, 15, 25, \ldots$. Find the value of the hundredth term.

1260. The probabilities of events X and Y are given in the following Venn diagram,



Represent the same information on a tree diagram, conditioned on X, finding and labelling all six branch probabilities.

1261. Prove by exhaustion that no square number ends in any of the digits $2, 3, 7, 8$.

1262. Make x the subject in $y=\dfrac{a\sqrt{x}+b}{c\sqrt{x}+d}$.

1263. Part of the *Diophantus identity* states that

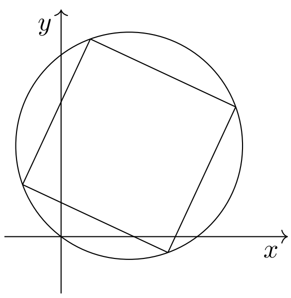
$$\left(a^2+b^2\right)\left(c^2+d^2\right)\equiv\left(ac-bd\right)^2+\left(ad+bc\right)^2.$$

Prove this result.

1264. A circle is given, for constants $a, b \in \mathbb{R}$, as

$$x^2+ax+y^2+by=0.$$

A square is inscribed, with all four of its vertices on the circumference.



Show that the square has area $2(a^2+b^2)$.

1265. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $x \in A$,
② $x \in A \cup B$.

1266. Write 4^{2x+3} in terms of 2^x.

1267. Three dice are rolled. State which, if either, of the following events has the greater probability:

- all scores are odd,
- no scores are odd.

1268. Simplify $\dfrac{1+\sqrt{\frac{1}{c}}}{1-\sqrt{\frac{1}{c}}}+\dfrac{1-\sqrt{\frac{1}{c}}}{1+\sqrt{\frac{1}{c}}}$.

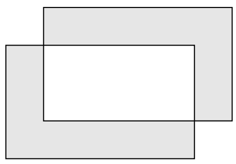
1269. Find c such that the curve $y=x^2+c$ satisfies the differential equation

$$\frac{dy}{dx}=\sqrt{4y-20}.$$

1270. A particle is projected horizontally, initial speed 5 ms^{-1}, from a point 1.6 metres above flat ground.

(a) Find the time of flight.
(b) Determine the angle below the horizontal at which the particle is travelling as it lands.

1271. Two identical rectangles are overlaid as depicted. Each has dimensions 12 cm $\times$ 8 cm, and the shaded area is 100 cm^2.



In each case, find the given quantity, or state that there is not enough information to find it:

(a) the area of the unshaded rectangle,
(b) the perimeter of the unshaded rectangle.

1272. Solve the equation $4\sqrt{x}\left(1+x\right)-\left(\sqrt{x}+1\right)^4=1$.

1273. In a convex polygon, every diagonal lies inside the polygon. Prove that the sum of the interior angles of a convex n-gon is $S=(n-2)\pi$ radians.

1274. Show that the following algebraic fraction cannot be simplified to a polynomial:

$$\frac{16x^4+5x^2-6x+1}{x-3}.$$

1275. Determine the distance from the centre of a cube of side length l to one of its vertices.

1276. Simplify $\{z \in \mathbb{R} : |z| \leq 5\} \setminus [2, \infty)$.

1277. The numbers p, $2p+1$, $2p^2-3$ are in arithmetic progression. Find all possible values of p.

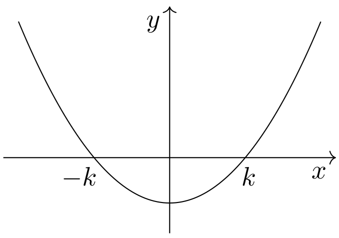
1278. Consider the graph $\log_2 x + \log_2 y = 1$.

(a) Show that the relationship between x and y is one of inverse proportion.

(b) Hence, sketch the graph.

1279. Show that $\lim_{h \to 0} \dfrac{(x+h)^4 - (x-h)^4}{2h} = 4x^3$.

1280. A quadratic graph $y = ax^2+bx+c$ is shown below, with its x axis intercepts marked:



State, with a reason, whether the following facts are necessarily true:

(a) “a is positive”,

(b) “b is zero”,

(c) “c is negative”.

1281. Two dice are rolled. Given that at least one shows a four, find the probability that both do.

1282. A symmetrical design is created by taking the curve $y = x^2$ and rotating it through 90°, 180°, and 270°, with centre of rotation at the origin. Points with either $|x|$ or $|y|$ greater than 2 are not drawn. Areas enclosed by the curves are then shaded.

(a) Sketch the design.

(b) Show that the design has area $\frac{4}{3}$.

1283. Show that $2x^2-x+1$ is not a factor of $2x^4-3x^3+2$.

1284. A triangle is drawn in a unit circle $x^2+y^2 = 1$, with its vertices on the circumference. Two of its sides are defined by the lines $x = 0$ and $y = \frac{3}{5}x-1$. Show that the other side is defined by the line $y = 1-\frac{5}{3}x$.

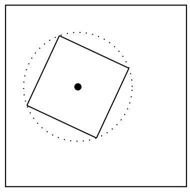
1285. Two distributions are given as

$$X \sim B(100, 1/2), \text{ and } Y \sim N(50, 25).$$

Show that $\mathbb{P}(X = 40) \approx \mathbb{P}(39.5 < Y < 40.5)$.

1286. Find b in simplified terms of a, if the quadratic x^2+4x+a has a factor of $x-b$.

1287. A unit square U has a smaller square S drawn somewhere inside it. S, when rotated around its centre, remains inside U. S has area a.



Find the set of possible values of a.

1288. Find the length of the line segment

$$x = p + t\cos\theta, \quad y = q + t\sin\theta, \quad t \in [-1, 1].$$

1289. If $x = \sin u$, show that $\left(\dfrac{du}{dx}\right)^2 = \tan^2 u + 1$.

1290. Two functions f and g are related by $f'(x) = g'(x)$.

(a) Show that $f(x) - g(x)$ is constant.

(b) A student claims “If $f'(x) = g'(x)$, then, for $a > 0$, the ratio $\log_a f(x) : \log_a g(x)$ doesn’t depend on x.” His proof is as follows:

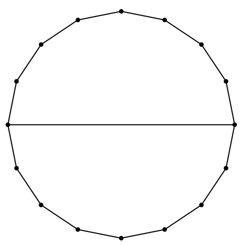
“Taking logs of both sides of $f(x) - g(x) = c$, we get $\log_a(f(x) - g(x)) = \log_a c$. The RHS is a constant which doesn’t depend on x; let’s call it k. Using a log rule on the LHS,

$$\frac{\log_a f(x)}{\log_a g(x)} = k.$$

Hence, the ratio $\log_a f(x) : \log_a g(x)$ doesn’t depend on x.”

Identify the error in the proof.

1291. A diameter, joining opposite vertices, is drawn on a regular 16-gon.



A pair of distinct vertices A, B is then chosen from among the remaining 14. Find the probability that the chord AB crosses the diameter.

1292. The interior angles of a quadrilateral form an AP. Give, in radians, the sum of the smallest and largest angles.

1293. The velocity of an object, for $t \geq 0$, is given by $v = \sqrt{t} - t$. At $t = 0$, the object is at the origin.

(a) Find the maximum velocity attained.

(b) Verify that the object returns to O at $t = \frac{16}{9}$.

1294. Show that points $(0, -2)$ and $(3, 0)$ are equidistant from the circle $x^2 + x + y^2 - 4y = 0$.

1295. Point A is on the line $y = 1 - 2x$, and $\overrightarrow{OA}$ is a unit vector. Determine the possible coordinates of A.

1296. State whether the following functions have a sign change at $x = 1$.

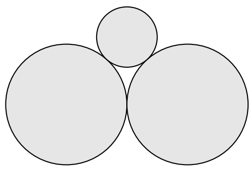
(a) $\mathrm{g}(x) = x(x-1)$,

(b) $\mathrm{g}(x) = x(x-1)^{-1}$,

(c) $\mathrm{g}(x) = x(x-1)^{-2}$.

1297. You are given that $\ln y = 3 \ln x + 4$. Describe the proportionality relationship between the variables y and x.

1298. Three circles of radius 5, 10 and 10 are all tangent to each other.



Show that all three can be contained within a circle of radius 20.

1299. The first four terms of an arithmetic sequence are given by $a, b, a+b, 8$. Find a and b.

1300. Prove that the sum of three consecutive cubes is divisible by 3.

——— End of 13th Hundred ———